Theoretical Design of Symmetrical Junction Stripline Circulators*

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Summary—**The results of detailed calculations based on Bosma's theory of 3-port stripline circulators are presented. The analysis has been modified to enable the scattering matrix, and therefore the circulator bandwidth, to be found. The equations of the 4-port junction have been derived and an expression for the scattering matrix of the general m-port junction is given. A comparison of experimental and theoretical results for the 3-port junction shows good agreement for circulators designed in the frequency band 2–3 Gc.**

I. Introduction

FERRITE junction circulators have in recent years found increasing use in microwave systems for both static and switching applications. Such circulators have been made in both waveguide and stripline form, and bandwidths of greater than 30 per cent have been achieved using 3-port junctions. The design procedure has been largely experimental and much time has been spent searching for ferrite configurations having suitable characteristics over the required frequency range.

The theoretical aspects of the stripline symmetrical 3-port junction circulator have been investigated in two papers by Bosma.[1,2] In the first of these, using simplified boundary conditions, a Fourier analysis produced two equations governing circulation which would normally require solution by a computer. Bosma then made assumptions which simplified these equations and derived an analytical expression of explicit form, which made calculation much easier. Unfortunately these approximations produced invalid results and so, in a second paper, the problem was reformulated.

In an equivalent treatment, using a Green's function, Bosma then produced equations more amenable to approximation. Valid solutions were obtained which agreed well with experiment. These approximations amount to considering only one of the many possible modes of circulation, and restricting the applied magnetic biasing field to values which produce small ratios of κ/μ (where μ and $\pm i\kappa$ denote the diagonal and off-diagonal components of the ferrite relative permeability tensor). The results therefore hold only for very small applied fields and for fields several times that required for ferromagnetic resonance. The latter condition is that used by Bosma in his experiments in the UHF band.

The object of the work described in this paper is to help to establish the validity of the equations governing circulation which were derived by Bosma.[1] Solutions of these equations are presented in Section III of this paper. In order to help in the interpretation of the results, and to extend the scope of Bosma's theory, the present authors have also derived the scattering matrix of the junction in Section II-B. This expression, within the limits of the approximate field analysis, holds for all real values of the permeability tensor elements μ and κ. In practice, the theory may be applied when the dissipative loss is small. Using the same approach the equations of the 4-port junction have been derived, and then the work has been extended to obtain an expression for the eigenvalues of the scattering matrix of the general m-port junction.

Since this work was completed before publication of Bosma[2] the notation and equations are from Bosma,[1] with a few modifications which are defined in the text.

II. Theory of the 3-Port Junction

A. Summary of Bosma's Theory

A symmetrical 3-port stripline junction is loaded with two ferrite disks. The inner conductors of the striplines are connected to the circumference of a central circular conducting plane which is of the same diameter as the disks. The ferrite fills the space between the central conductor and the outer ground planes, see Fig. 1. It is assumed that the ferrite, dielectric and conducting materials are loss-free.

A uniform magnetic field is applied in a direction perpendicular to the plane of the disks. Only solutions for waves with electric fields purely in the z direction and magnetic fields purely in the plane of the disks are considered. The fields are independent of z and have time dependence $\exp(-i\omega t)$.

The boundary conditions are simplified by assuming $H_\phi(R, \phi)=0$, R being the radius of the disks, except over the width of the stripline connections. At these regions it is assumed that

$$H_\phi = \begin{cases} A \text{ at port 1} \\ B \text{ at port 2} \\ C \text{ at port 3} \end{cases}$$

where A, B and C are complex constants. Bosma[1] shows that this is the only boundary condition which need be

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[1] H. Bosma, "On the principle of stripline circulation," *Proc. IEE*, vol. 109, Pt. B, Suppl. No. 21, pp. 137–146; January, 1962.

[2] H. Bosma, "On stripline Y-circulation at UHF," to be published in IEEE Trans. on Microwave Theory and Techniques, Vol. MTT-12, January, 1964.

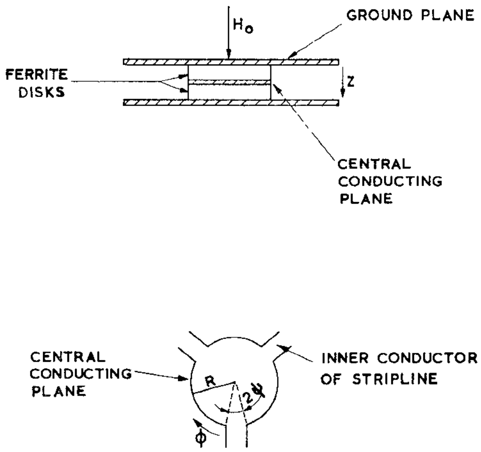

Fig. 1—Schematic diagram of 3-port circulator.

retained. A Fourier analysis of $H_\phi(R, \phi)$ is then made and, by comparison with the corresponding series obtained from Maxwell's equations, $E_z(\rho, \phi)$ is found. On averaging $E_z(R, \phi)$ over each of the three ports the following equations result:

$$E_z^{\mathrm{I}} = K(C - B) + iL(A + B + C) + iM\left(A - \frac{B + C}{2}\right) \tag{1}$$

$$E_z^{\mathrm{II}} = K(A - C) + iL(A + B + C) + iM\left(B - \frac{C + A}{2}\right) \tag{2}$$

$$E_z^{\mathrm{III}} = K(B - A) + iL(A + B + C) + iM\left(C - \frac{A + B}{2}\right) \tag{3}$$

where

$$K = \frac{\sqrt{3}}{2} \sum_{p'=0}^{\infty} (K_{3p'+1} - K_{3p'+2})$$

$$L = \sum_{p'=0}^{\infty} L_{3p'}$$

$$M = \sum_{p'=0}^{\infty} (L_{3p'+1} + L_{3p'+2})$$

and the terms are given by

$$L_0 = \frac{Z_{\mathrm{eff}} \psi J_0(x)}{\pi J_0'(x)}$$

$$L_n = \frac{2Z_{\mathrm{eff}} \sin^2 (n\psi) J_n(x) J_n'(x)}{n^2 \pi \psi \left[\left(\frac{\kappa}{\mu} \frac{n}{x} J_n(x)\right)^2 - (J_n'(x))^2\right]}$$

$$K_n = -\frac{\kappa}{\mu} \frac{n}{x} \frac{J_n(x)}{J_n'(x)} L_n.$$

To complete the list of definitions we have

$$Z_{\mathrm{eff}} = \frac{1}{Y_{\mathrm{eff}}} = \sqrt{\frac{\mu_0 \mu_{\mathrm{eff}}}{\epsilon_0 \epsilon}}$$

$$x = kR$$

$$k^2 = \omega^2 \mu_0 \epsilon_0 \mu_{\mathrm{eff}} \epsilon$$

ϵ = relative permittivity of ferrite

$$\mu_{\mathrm{eff}} = \frac{\mu^2 - \kappa^2}{\mu}$$

2ψ = angular width of stripline in radians.

Field matching at the ferrite-dielectric interface is achieved by equating the wave impedances of the ferrite and stripline media at the three ports.

The junction will act as a circulator with transmission from port 1 to port 2, if and only if

$$\frac{E_z^{\mathrm{II}}}{B} = \frac{E_z^{\mathrm{III}}}{C} = -Z_d \tag{4}$$

and $C = 0$ where Z_d is the intrinsic impedance of the medium filling the stripline. These restrictions give

$$p = \frac{-q(3r^2 - q^2)}{(r^2 + q^2)} \tag{5}$$

$$\frac{Z_{\mathrm{eff}}}{Z_d} = \frac{(r^2 + q^2)}{(3q^2 - r^2)r} \tag{6}$$

where $p = Y_{\mathrm{eff}}(L + M)$, $q = Y_{\mathrm{eff}}(L - M/2)$, $r = Y_{\mathrm{eff}} K$.

Eqs. (5) and (6) define a perfect circulator and it can easily be verified that under these conditions the junction is matched. By some algebra, (5) and (6) may be written in alternative forms which prove useful. Eq. (5) becomes

$$3q^2 = r^2\left(1 - \frac{8l}{m}\right) \tag{7}$$

where $l = Y_{\mathrm{eff}} L$, $m = Y_{\mathrm{eff}} M$. Eq. (6) becomes

$$\frac{Z_{\mathrm{eff}}}{Z_d} = \frac{q}{3rl} \cdot \tag{8}$$

B. Scattering Matrix Analysis

The scattering matrix is defined by

$$\boldsymbol{b} = S\boldsymbol{a}. \tag{9}$$

$\boldsymbol{a}$ and $\boldsymbol{b}$ are column vectors with elements $a_{r'}$ and $b_{r'}$ ($r' = 1, 2, 3$). The $a_{r'}$ element represents the amplitude of the wave entering the rth port and $b_{r'}$ that of the wave leaving the port where both are measured at the terminal plane. These reference planes are taken to be at the ferrite-dielectric interface. For a symmetrical 3-port junction the scattering matrix has eigenvectors $\boldsymbol{u}_j$ ($j = 0, 1, 2$). These are column vectors with elements $\exp(\frac{2}{3}\pi i r' j)$. If the corresponding eigenvalues are λ_j, the scattering matrix is

$$S = \begin{bmatrix} s_0 & s_1 & s_2 \\ s_2 & s_0 & s_1 \\ s_1 & s_2 & s_0 \end{bmatrix} \tag{10}$$

where

$$s_j = \frac{1}{3} \sum_{q'=0}^{2} \lambda_{q'} \exp(-2\pi i j q'/3). \tag{11}$$

Since the system is loss-less, S is unitary and the eigenvalues have unit amplitude.

Consider the characteristic equation

$$S\boldsymbol{u}_j = \lambda_j \boldsymbol{u}_j. \tag{12}$$

By comparison with (9) it can be seen that $\boldsymbol{u}_j$ represents a possible field excitation in the junction with the fields at the terminal planes proportional to the elements of the eigenvector and λ_j represents a reflection coefficient measured at each terminal plane. Now consider the fields throughout the junction due to the excitation of one particular eigenvector $\boldsymbol{u}_j$ of the scattering matrix. If edge effects are ignored the fields at the terminal planes may be written as

$$E_z = (a+b)\sqrt{Z_d} \tag{13}$$

$$H_\phi = (a-b)\sqrt{\frac{1}{Z_d}} \cdot \tag{14}$$

Again, a and b represent ingoing and outgoing waves with the same ratio λ_j at each port. Thus the wave impedance is

$$\frac{E_z}{H_\phi} = Z_d \left\{ \frac{1+\frac{b}{a}}{1-\frac{b}{a}} \right\} = Z_d \left\{ \frac{1+\lambda_j}{1-\lambda_j} \right\}. \tag{15}$$

Defining $\lambda_j = \exp(i\theta_j)$, the wave impedance becomes

$$\frac{E_z}{H_\phi} = iZ_d \cot \frac{\theta_j}{2} \cdot \tag{16}$$

The eigenvectors of the junction are

$$\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix}$$

where 1, ω, ω^2 are the cube roots of unity.

Considering each eigensolution in turn, E_z and H_ϕ are taken to be proportional to the eigenvector elements. Using (1), (2) and (3) the following equations result:

for $j = 0$, $\quad A:B:C = 1:1:1$

therefore

$$\frac{E_z}{H_\phi} = 3iL = 3ilZ_{\text{eff}} \tag{17}$$

for $j = 1$, $\quad A:B:C = 1:\omega:\omega^2$

therefore

$$\frac{E_z}{H_\phi} = i\left(\frac{3}{2}m - \sqrt{3}\,r\right) Z_{\text{eff}} \tag{18}$$

for $j = 2$, $\quad A:B:C = 1:\omega^2:\omega$

therefore

$$\frac{E_z}{H_\phi} = i\left(\frac{3}{2}m + \sqrt{3}\,r\right) Z_{\text{eff}}. \tag{19}$$

By virtue of (16) the following equations result for the arguments of the eigenvalues:

$$\cot \frac{\theta_0}{2} = 3l \frac{Z_{\text{eff}}}{Z_d} \tag{20}$$

$$\cot \frac{\theta_1}{2} = \left(\frac{3}{2}m - \sqrt{3}\,r\right) \frac{Z_{\text{eff}}}{Z_d} \tag{21}$$

$$\cot \frac{\theta_2}{2} = \left(\frac{3m}{2} + \sqrt{3}\,r\right) \frac{Z_{\text{eff}}}{Z_d} \tag{22}$$

where

$$\frac{Z_{\text{eff}}}{Z_d} = \sqrt{\frac{\mu_{\text{eff}}\epsilon_d}{\mu_d \epsilon}}$$

and μ_d, ϵ_d relate to the dielectric. It is useful to write these equations in terms of the series because of the simplification which occurs,

$$\cot \frac{\theta_0}{2} = 3 \frac{Z_{\text{eff}}}{Z_d} \sum_{n=3p'}^{\infty} \frac{\epsilon_n \sin^2(n\psi) J_n(x) J_n'(x)}{n^2 \pi \psi \left[\left(\frac{\kappa}{\mu} \frac{n}{x} J_n(x) \right)^2 - (J_n'(x))^2 \right]} \tag{23}$$

where

$$p' = 0, 1, 2, \cdots \quad \text{and} \quad \epsilon_n = \begin{cases} 1 & \text{for } n = 0 \\ 2 & \text{for } n \neq 0 \end{cases}$$

$$\cot \frac{\theta_1}{2} = 3 \frac{Z_{\text{eff}}}{Z_d} \left\{ \sum_{n=3p'+1}^{\infty} \frac{\sin^2(n\psi) J_n(x)}{n^2 \pi \psi \left[\left(\frac{\kappa}{\mu} \frac{n}{x} J_n(x) - J_n'(x) \right) \right]} - \sum_{n=3p'+2}^{\infty} \frac{\sin^2(n\psi) J_n(x)}{n^2 \pi \psi \left[\left(\frac{\kappa}{\mu} \frac{n}{x} J_n(x) + J_n'(x) \right) \right]} \right\} \tag{24}$$

$$\cot \frac{\theta_2}{2} = 3 \frac{Z_{\text{eff}}}{Z_d} \left\{ - \sum_{n=3p'+1}^{\infty} \frac{\sin^2(n\psi) J_n(x)}{n^2 \pi \psi \left[\left(\frac{\kappa}{\mu} \frac{n}{x} J_n(x) + J_n'(x) \right) \right]} + \sum_{n=3p'+2}^{\infty} \frac{\sin^2(n\psi) J_n(x)}{n^2 \pi \psi \left[\left(\frac{\kappa}{\mu} \frac{n}{x} J_n(x) - J_n'(x) \right) \right]} \right\}. \tag{25}$$

Having found the eigenvalues of the scattering matrix, the condition for circulation,[3] *i.e.*, that the eigenvalues must be spaced at 120° intervals around the unit circle, gives two equations equivalent to (5) and (6). However, the formulation of these equations is such that they remove many of the difficulties caused by singularities, both in interpretation and computation, of solutions to (5) and (7). In addition, by using (10) to derive the scattering matrix, the performance can be computed of any particular junction, as a function of frequency.

III. Computations

A. Method

In order to obtain practical design data from the theory it is necessary to solve (5) and (6). This has been done by making x a parameter and solving (5) for κ/μ.

The associated values of p, q and r have then been substituted in (6) and Z_{eff}/Z_d computed.

In terms of the junction parameters (5) is implicit in form, while (6) is explicit, so only the former presents a problem. It may be rewritten as

$$r^2 = \frac{q^2(q-p)}{3q+p}\cdot \tag{26}$$

Defining a new series s by $r=\sqrt{3}/2\cdot\kappa/\mu\cdot s$, (26) becomes

$$\left(\frac{\kappa}{\mu}\right)^2 = \frac{4}{3}\,\frac{q^2(q-p)}{s^2(3q+p)} \tag{27}$$

and so is of the form

$$\left(\frac{\kappa}{\mu}\right)^2 = f\left\{\left(\frac{\kappa}{\mu}\right)^2,\ x,\ \psi\right\}.$$

For given x and ψ, this has been solved by an iterative method due to Wegstein.[4]

B. Singularities of the Problem

The magnitude of individual terms of the series becomes infinite when

$$\frac{\kappa}{\mu}\cdot\frac{n}{x}\cdot J_n(x) \pm J_n'(x) = 0 \qquad (n = 0, 1, 2, \cdots) \tag{28}$$

and, from the definitions above, when $n=1, 2, 4, 5 \cdots$ then

$$\lim \frac{K_n}{L_n} = \pm 1. \tag{29}$$

For these values of n

$$\lim \frac{r}{m} = \lim \frac{K}{M} = \begin{cases} \mp \dfrac{\sqrt{3}}{2} & \text{for } n = 3p'+2 \\ \pm \dfrac{\sqrt{3}}{2} & \text{for } n = 3p'+1 \end{cases} \qquad p = 0, 1, 2, \cdots.$$

Since l remains finite for these values of n,

$$\lim \frac{r}{m} = -\lim \frac{r}{2q}, \tag{30}$$

by definition of q. Therefore $\lim r^2/3q^2 = 1$.

Using this result, it can be seen from (7), which is equivalent to (5), that the singularities expressed by (28) for these values of n appear to satisfy the equation in the sense that both sides approach infinity together. Eq. (8) shows that a finite value of Z_{eff}/Z_d is obtained from these "solutions." During computations such results appeared, characterized by very high values of p, q and r. That they do not represent "true" solutions, in general, may be seen by considering (23), (24) and (25). These show that the singularities occur when the appropriate eigenvalue argument approaches zero. As this imposes no restriction on the spacing of the eigenvalues, the two conditions have still to be satisfied for circulator action, and so the singular solutions to (5) must be completely spurious.

C. Results

In order to keep the computing time within an economic level, solutions were sought only for $\psi = 20°$ and in the region $1.80 \le x \le 3.46$, $0 < \kappa/\mu < 0.825$. The series were restricted by considering only terms for which $n \le 6$.

Solutions of (5) are shown in Fig. 2, plotted as x against κ/μ. The dashed curves superimposed on this plot are solutions to (28) for $n=1$, 2 and 3. Corresponding to any solution of (5), (6) [or (8)] explicitly gives values of Z_{eff}/Z_d against κ/μ ,which are plotted in Figs. 3, 4 and 5.

It should be noted that negative values of Z_{eff}/Z_d, as defined by (6) or (8), merely imply circulation in the opposite sense to that for positive values, for a given direction of applied magnetic biasing field. Physically Z_{eff}/Z_d would, of course, always be positive.

D. Discussion

Fig. 2 shows several modes of operation, and these have been labelled arbitrarily as 1, 1A, 2 and 3. The curves appear to stem from the points A, B, C, D and E which all define double singularities. Moreover, with the exception of mode 1A, the curves are in the vicinity of, and cross over, the loci of the singularities.

[3] B. A. Auld, "The synthesis of symmetrical waveguide circulators," IRE Trans. on Microwave Theory and Techniques, vol. MTT-7, pp. 238–246; April, 1959.
[4] G. N. Lance, "Numerical Methods for High-Speed Computers," Iliffe and Sons, Ltd., London, England, pp. 134–138; 1960.

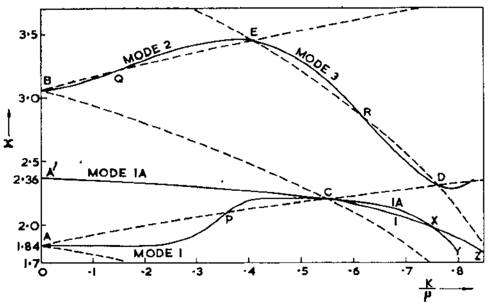

Fig. 2—Solutions of the first circulation equation.

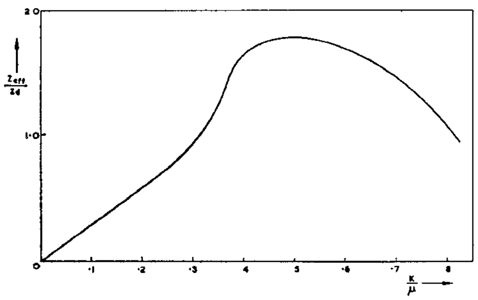

Fig. 3—Solutions of the second circulation equation, mode 1.

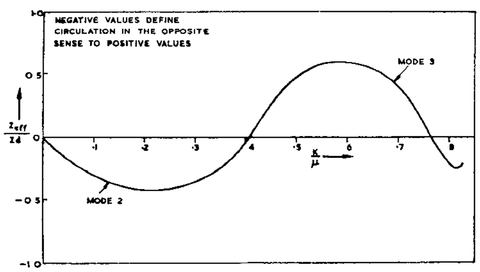

Fig. 4—Solutions of the second circulation equation, modes 2 and 3.

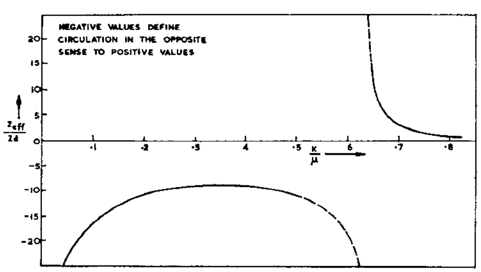

Fig. 5—Solutions of the second circulation equation, mode 1A.

From (20), (21) and (22) it can be seen that the points A, B, D and E would represent two-fold degeneracies of the eigenvalues for nonzero values of Z_{eff}/Z_d. The degeneracy is avoided if $Z_{eff}/Z_d=0$, and that this is the case for circulation to be possible is shown by Figs. 3 and 4. Clearly, though, points A and B cannot represent circulators since the junction becomes reciprocal for $\kappa/\mu=0$. In addition, points D and E cannot represent circulators because this would require that $\mu_{eff}=0$, and this is only true when $\kappa=\mu$. However, it is possible, at least in principle, to find circulator solutions arbitrarily close to these points.

The point C is a special case, in this region, in that the two singular terms occur in the one expression of (25) and so do not imply an eigenvalue degeneracy. The computations show that the two curves passing through C produce two different sets of values of Z_{eff}/Z_d, p, q and r near this point. Thus it appears that the double singularity is associated with two separate limits which depend on the path used to approach C. It will also be noted that the directions of circulation for modes 1 and 1A are opposite in the vicinity of point C. In this region the scattering matrix changes rapidly with small variations of frequency and disk radius, and so the neighborhood of C is undesirable for design purposes.

E. Design Procedure

To use these curves for the design of circulators it is necessary to find values of Z_{eff}/Z_d which are feasible. Curves of μ_{eff} against κ/μ are plotted using the Polder tensor formulation.[5] The intersection of such curves with those derived from Figs. 3, 4 and 5 may be found for suitable values of ϵ_d/ϵ. From the values of μ_{eff} and κ/μ obtained at an intersection, x may be found from Fig. 2. Finally, for a given frequency, and for an appropriate ferrite-saturation magnetization, the radius of the disks may be calculated.

Using such a procedure, it was found that the values of Z_{eff}/Z_d for mode 1A were too high for practical use, except for values of κ/μ near 0.75. Modes 1, 2 and 3 provided useful design data under a variety of conditions, with applied fields both above and below the value required for ferromagnetic resonance.

From mode 1, Bosma's approximation[2] that $x=1.84$ can be seen to be very good in the range $0<\kappa/\mu<0.25$. From Fig. 3, over the same range, for $\psi=20°$ there results the further approximation $Z_{eff}/Z_d \sim 3\kappa/\mu$.

F. Phase Shift

For perfect circulation from port 1 to port 2, (10) and (11) show that

$$\left.\begin{array}{l} s_0 = s_1 = 0 \\ s_2 = \exp(i\theta_0) \end{array}\right\}. \tag{31}$$

θ_0 is the argument of the zero-order eigenvalue and from (31) is the phase shift in transmission through the circulator. The phase shift is determined by (20) for all modes. Using (8) the phase shift at circulation may be written as

$$\theta_0 = 2 \cot^{-1} \frac{q}{r}. \tag{32}$$

The shifts for the modes are plotted in Fig. 6. It may be noted that phase shifts of $\pm 180°$, $\pm 120°$ and $0°$ correspond to the intersections between "circulation" and "singularity" curves shown in Fig. 2.

[5] B. Lax and K. J. Button, "Microwave Ferrites and Ferromagnetics," McGraw-Hill Book Company, Inc., New York, N. Y., pp. 151–155; 1962.

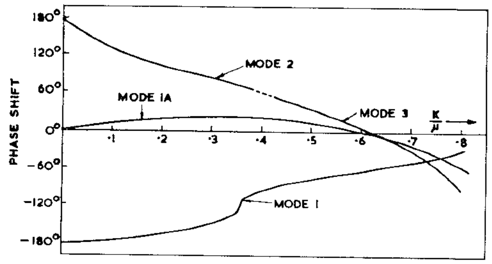

Fig. 6—Phase shift of transmission at circulation.

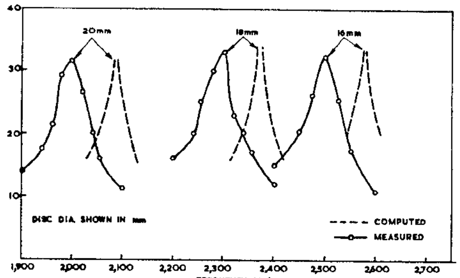

Fig. 7—Comparison of experimental and theoretical results for circulator designs based on mode 3.

G. *Accuracy of Computations*

The error in the results, due to restricting the number of terms in the series, was estimated to be about 2 per cent in the value of x for a given κ/μ.

IV. Experimental Verification

For mode 1 Bosma[2] found good agreement between the predicted and measured performance. These experiments were conducted in the UHF band with applied biasing fields sufficiently above ferromagnetic resonance to produce small values of κ/μ. He has also verified that circulation in mode 2 is in the opposite direction to that in mode 1.[6]

Measurements have been made on circulators whose design was based on mode 3. The performance of devices using disks of Ferroxcube 5C1 in the range 2–3 Gc was calculated and measured. In each case the measured frequency for best circulation was within 100 Mc of the predicted value. The average frequency error was about 4 per cent. The predicted bandwidth ($\sim$3 per cent) was in all cases smaller than that measured ($\sim$5 per cent). A typical comparison of theoretical and experimental curves is shown in Fig. 7.

V. Theory of Junctions with More Than Three Ports

A. *The 4-Port Junction*

The symmetrical 4-port junction may be analyzed in the same fashion as the 3-port. Because of the practical interest of these junctions the results of such an analysis are recorded here, although no attempt has yet been made to obtain numerical solutions. Since, in general, the 4-port circulator requires the adjustment of three physical parameters it may be necessary to introduce an extra variable, for instance by the insertion of a metal or dielectric pin into the ferrite disks. The equations quoted below are for the simple configuration without such an extra parameter, but they can be modified to include different cases.

[6] H. Bosma, private communication.

The ports are now spaced at 90° intervals around the circumference of the ferrite disks, and the RF magnetic fields at the ports are taken to be A, B, C and D. The average electric fields at the ports become

$$E_z^{\mathrm{I}} = K(D - B) + iLA + i(B + D)M + iCN \quad (33)$$
$$E_z^{\mathrm{II}} = K(A - C) + iLB + i(C + A)M + iDN \quad (34)$$
$$E_z^{\mathrm{III}} = K(B - D) + iLC + i(D + B)M + iAN \quad (35)$$
$$E_z^{\mathrm{IV}} = K(C - A) + iLD + i(A + C)M + iBN \quad (36)$$

where K_n, L_n, and L_0 are as for the 3-port case, and

$$K = \sum_{p'=0}^{\infty} (K_{4p'+1} - K_{4p'+3})$$
$$L = \sum_{p'=0}^{\infty} L_{p'}$$
$$M = \sum_{p'=0}^{\infty} (L_{4p'} - L_{4p'+2})$$
$$N = \sum_{p'=0}^{\infty} (L_{2p'} - L_{2p'+1}).$$

Matching the wave impedances gives

$$\frac{E_z^{\mathrm{II}}}{B} = \frac{E_z^{\mathrm{III}}}{C} = \frac{E_z^{\mathrm{IV}}}{D} = -Z_d \quad (37)$$

and imposing the conditions $C = D = 0$ produces

$$N^2 = K^2 + M^2 \quad (38)$$
$$MZ_d = K(L + N) \quad (39)$$
$$KZ_d = M(L - N). \quad (40)$$

By some algebraic manipulation, (39) and (40) can be rearranged to the form

$$N = \frac{2KM}{Z_d} \quad (41)$$
$$NL = M^2 - K^2. \quad (42)$$

Substitution in (33) verifies that the circulator is matched.

Eqs. (38), (41) and (42) define the circulator action. Normalizing, by putting $n = Y_{\mathrm{eff}}N$, $m = Y_{\mathrm{eff}}M$, $r = Y_{\mathrm{eff}}K$ and $l = Y_{\mathrm{eff}}L$, the equations finally become

$$n^2 = m^2 + r^2 \quad (43)$$
$$nl = m^2 - r^2 \quad (44)$$
$$\frac{Z_{\mathrm{eff}}}{Z_d} = \frac{n}{2rm}. \quad (45)$$

Eqs. (43) and (44) are implicit and their simultaneous solution involves a two-dimensional iterative process. Consequently, the increase in computing time, compared with the 3-port case, will be substantial.

The scattering matrix analysis is straightforward, and using the eigenvectors

$$u_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \quad u_1 = \begin{Bmatrix} 1 \\ i \\ -1 \\ -i \end{Bmatrix} \quad u_2 = \begin{Bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{Bmatrix} \quad u_3 = \begin{Bmatrix} 1 \\ -i \\ -1 \\ i \end{Bmatrix},$$

the eigenvalue arguments are given by

$$\cot\frac{\theta_0}{2} = \frac{Z_{\text{eff}}}{Z_d}(l + n + 2m) \tag{46}$$

$$\cot\frac{\theta_1}{2} = \frac{Z_{\text{eff}}}{Z_d}(l - n - 2r) \tag{47}$$

$$\cot\frac{\theta_2}{2} = \frac{Z_{\text{eff}}}{Z_d}(l + n - 2m) \tag{48}$$

$$\cot\frac{\theta_3}{2} = \frac{Z_{\text{eff}}}{Z_d}(l - n + 2r). \tag{49}$$

The full series are written as

$$\cot\frac{\theta_0}{2} = 4\frac{Z_{\text{eff}}}{Z_d}\sum_{n=4p'}^{\infty} \cdot \frac{\epsilon_n \sin^2(n\psi)J_n(x)J_n'(x)}{n^2\pi\psi\left[\left(\frac{\kappa}{\mu}\frac{n}{x}J_n(x)\right)^2 - (J_n'(x))^2\right]} \tag{50}$$

summed over

$$p' = 0, 1, 2, \cdots \qquad \epsilon_n = \begin{cases} 1 & \text{for } n = 0 \\ 2 & \text{for } n \neq 0 \end{cases}$$

$$\cot\frac{\theta_1}{2} = 4\frac{Z_{\text{eff}}}{Z_d}\left\{\sum_{n=4p'+1}^{\infty}\frac{\sin^2(n\psi)J_n(x)}{n^2\pi\psi\left[\left(\frac{\kappa}{\mu}\frac{n}{x}J_n(x) - J_n'(x)\right)\right]} - \sum_{n=4p'+3}^{\infty}\frac{\sin^2(n\psi)J_n(x)}{n^2\pi\psi\left[\left(\frac{\kappa}{\mu}\frac{n}{x}J_n(x) + J_n'(x)\right)\right]}\right\} \tag{51}$$

$$\cot\frac{\theta_2}{2} = 4\frac{Z_{\text{eff}}}{Z_d}\sum_{n=4p'+2}^{\infty} \cdot \frac{2\sin^2(n\psi)J_n(x)J_n'(x)}{n^2\pi\psi\left[\left(\frac{\kappa}{\mu}\frac{n}{x}J_n(x)\right)^2 - (J_n'(x))^2\right]} \tag{52}$$

$$\cot\frac{\theta_3}{2} = 4\frac{Z_{\text{eff}}}{Z_d}\left\{-\sum_{n=4p'+1}^{\infty}\frac{\sin^2(n\psi)J_n(x)}{n^2\pi\psi\left[\left(\frac{\kappa}{\mu}\frac{n}{x}J_n(x) + J_n'(x)\right)\right]} + \sum_{n=4p'+3}^{\infty}\frac{\sin^2(n\psi)J_n(x)}{n^2\pi\psi\left[\left(\frac{\kappa}{\mu}\frac{n}{x}J_n(x) - J_n'(x)\right)\right]}\right\}. \tag{53}$$

B. The m-Port Junction

An analysis of the m-port junction gives the following expression for the argument of the jth eigenvalue:

$$\cot\frac{\theta_j}{2} = \frac{Z_{\text{eff}}}{Z_d}\frac{m\psi}{\pi}\sum_{n=mp'+j}\left(\frac{\sin n\psi}{n\psi}\right)^2 \cdot \frac{J_n(x)}{\left[\frac{\kappa}{\mu}\frac{n}{x}J_n(x) - J_n'(x)\right]} \tag{54}$$

summed over $p' = 0, \pm 1, \pm 2, \cdots$.

The elements of the scattering matrix can then be derived from

$$s_j = \frac{1}{m}\sum_{k'=0}^{m-1}\exp i(\theta_j - 2\pi jk'/m). \tag{55}$$

VI. Conclusions

Over an interesting region of operation, a set of solutions has been obtained to the 3-port circulator equations derived by Bosma. These solutions, combined with a complementary approach to the problem which allows the scattering matrix to be derived, have enabled designs for 3-port circulators to be made. The agreement between experimental and theoretical results, which was referred to in Section IV, certainly supports the validity of the analysis.

A basis now exists on which to extend an investigation into more complicated junctions with, say, composite loading or four ports. To this end the relevant equations of the 4-port junction have been quoted, and only minor modifications are required to allow for an additional variable (*i.e.*, an extra adjustable physical parameter) to be introduced. It is quite possible that further computations on the 3-port junction, with the addition of such an extra parameter, will lead to some insight into the design of broad-band circulators.

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